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Nonstandard Methods in Quantum Field Theory I: A Hyperfinite Formalism of Scalar Fields

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A nonstandard approach to axiomatic quantum field theory is given. Nonstandard axioms for a Hermitian scalar field is proposed, where the field operators act on a hyperfinite-dimensional Hilbert space. The axioms are shown to be equivalent to the Gårding–Wightman axioms. An example of a model of the nonstandard axioms is examined.

KEY WORDS: Wightman fields; canonical commutation relations; quantum fields; nonstandard analysis; axiomatic field theory.

1. INTRODUCTION

Axiomatic quantum field theory began with the Gårding–Wightman axioms (Gårding and Wightman, 1964; Reed and Simon, 1975), which are written in terms of field operators in a Hilbert space. The Gårding–Wightman axioms are reformulated as the Wightman axioms (Streater and Wightman, 1964) in terms of tempered distributions, and then Euclideanized by Osterwalder and Schrader (1973). The aim of this paper is to give another reformulation of the Gårding–Wightman axioms in terms of the operators on a *hyperfinite-dimensional linear space*, which is constructed in *nonstandard analysis*, originated by Robinson (1966). This reformulation uses our recent mathematical results on nonstandard linear operators (Yamashita and Ozawa, 2001).

Kelemen and Robinson (1972), who suggested a nonstandard method of constructing the $\lambda : \phi_2^4(x) :$ model, claimed the need of the nonstandard axioms of quantum field theory. However, the subject has not been studied after Kelemen and Robinson, while several authors (e.g., Albeverio *et al.*, 1986; Gudder, 1994; Nakamura, 1991, 1997; Ojima and Ozawa, 1993; Ozawa, 1997; Yamashita, 1998; Yamashita and Ozawa, 2000) have attempted to apply nonstandard analysis to quantum physics.

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We first give some mathematical preliminary notions; the *standard part* of an internal bounded operator and the *nonstandard representations* of the canonical commutation relation (CCR). In this paper, we shall treat the CCR as relations satisfied by unbounded operators, called creation and annihilation operators, so that our treatment can be adjusted to the Gårding–Wightman axioms for scalar fields in which unbounded operators are employed. Nonstandard representations of the CCR in terms of bounded operators (the CCR of the Weyl form) were examined by Yamashita and Ozawa (2000).

In Section 5, the nonstandard axioms for neutral scalar fields are given, and proven to be equivalent to the Gårding–Wightman axioms. Then, we give an example of a free field satisfying the axioms, using the nonstandard representations of the CCR mentioned above.

Other quantum fields, especially gauge fields, will be studied in the subsequent papers.

2. NONSTANDARD ANALYSIS

We follow the superstructure approach to nonstandard analysis (Chang and Keisler, 1990; Hurd and Loeb, 1985). Their basic definitions were reviewed in Yamashita and Ozawa (2000).

Instead of $*$, we let $\star$ denote the ordinary star-map in nonstandard analysis. For a set S, let ${}^\sigma S = \{{}^\star s \mid s \in S\}$. We identify ${}^\star z$ with z for all $z \in \mathbf{C}$, i.e., we assume the base set of the standard superstructure contains $\mathbf{C}$. So ${}^\sigma S = S$ if S is a subset of $\mathbf{C}$, e.g., ${}^\sigma\mathbf{C} = \mathbf{C}$, ${}^\sigma\mathbf{R} = \mathbf{R}$, ${}^\sigma\mathbf{Z} = \mathbf{Z}$, and ${}^\sigma\mathbf{N} = \mathbf{N}$. Let $\mathbf{R}^+$, ${}^\star\mathbf{R}_0$, ${}^\star\mathbf{R}_0^+$, ${}^\star\mathbf{R}_\infty^+$ and ${}^\star\mathbf{N}_\infty$ denote the set of positive real numbers, infinitesimal $\star$-real numbers, positive infinitesimal $\star$-real numbers, positive infinite $\star$-real numbers and infinite $\star$-natural numbers respectively. It is shown that ${}^\star\mathbf{N}_\infty = {}^\star\mathbf{N}\backslash\mathbf{N}$. We write $x > \infty$ if $x \in \mathbf{R}_\infty^+$, and $x < \infty$ if $x \in \operatorname{fin}{}^\star\mathbf{R}^+ = {}^\star\mathbf{R}^+\backslash{}^\star\mathbf{R}_\infty^+$. If $r \in {}^\star\mathbf{R}$ and $|r| < \infty$, ${}^\circ r$ (or $\operatorname{st}(r)$) denotes the standard part of r. If $r > \infty$, we write ${}^\circ r = \infty$.

Let $x, y \in {}^\star\mathbf{R}^+$. We say x is of the *order* of y, $x \asymp y$, iff $x/y < \infty$ and $y/x < \infty$. We write $x \ll y$ if $x/y \approx 0$.

For a hyperfinite ($\star$-finite) set F, let $|F|$ denote the internal cardinal number of F ($|F| \in {}^\star\mathbf{N}$).

Let $(X, \mathcal{O})$ be a topological space, and $\mathcal{O}_x$ the system of open neighborhood of $x \in X$. The *monad* of $x \in X$ is the subset $\operatorname{mon}_\mathcal{O}(x) = \bigcap {}^\star O (O \in \mathcal{O}_x)$ of ${}^\star X$. The set of *near standard* points is the set $\operatorname{ns}({}^\star X) = \bigcup \operatorname{mon}_\mathcal{O}(x)(x \in X)$. It is shown that $(X, \mathcal{O})$ is Hausdorff iff $x \neq y \Rightarrow \operatorname{mon}_\mathcal{O}(x) \cap \operatorname{mon}_\mathcal{O}(y) = \phi$. Thus for the Hausdorff space $(X, \mathcal{O})$, we can define the equivalence relation $\overset{\mathcal{O}}{\approx}$ on $\operatorname{ns}{}^\star X$ by $a \overset{\mathcal{O}}{\approx} b$ iff for some $x \in X$ $a \in \operatorname{mon}_\mathcal{O}(x)$ and $b \in \operatorname{mon}_\mathcal{O}(x)$.

Let $(X, \|\cdot\|)$ be an internal normed linear space. Define the relation $\approx$ on ${}^\star X$ by $x \approx y$ iff $\|x - y\| \approx 0$. The *principal galaxy* of ${}^\star X$ is the set $\operatorname{fin}({}^\star X) = \{x \in {}^\star X \mid \|x\| < \infty\}$. Let $\hat{X}$ denote the equivalence classes of $\operatorname{fin}({}^\star X)$ under the equivalence relation $\approx$. For $x \in \operatorname{fin}(X)$, let $\hat{x}$ be the equivalence class

$\hat{x} = \{y \in {}^\star X \mid x \approx y\}$. Clearly $\hat{X} = \{\hat{x} \mid x \in \mathrm{fin}({}^\star X)\}$. Define the norm $\|\cdot\|$ on $\hat{X}$ by $\|\hat{x}\| = {}^\circ\|x\|$ (well-defined). The pair$(\hat{X}, \|\cdot\|)$ turns out to be a Banach space, called the *standardization* of $(X, \|\cdot\|)$, in a κ-saturated model with $\kappa > \aleph_0$. In a similar way, the standardization is defined for an internal pre-Hilbert space $(X, \langle,\rangle)$, and it becomes a Hilbert space.

For a (standard) normed linear space $(X, \|\cdot\|)$, we abbreviate $\widehat{{}^\star X}$ to $\hat{X}$. In this case, the Banach space $(\hat{X}, \|\cdot\|)$ is called the *nonstandard hull* of $(X, \|\cdot\|)$.

Let X be an internal normed linear space, and $S \subseteq X$. Define $\tilde{S} \subseteq X$ by $\tilde{S} = \{\xi \in X \mid$ for any $\epsilon \in \mathbf{R}^+$, there is $\eta \in S$ such that $\|\xi - \eta\| < \epsilon\}$. If $S \subseteq \mathrm{fin}(X)$, we find $\tilde{S} = \cup\hat{S} = \{\xi \in X \mid \hat{\xi} \in \hat{S}\}$. In this paper, an external set $S \subset \mathrm{fin} X$ is called *closed* if $S = \cup S'$ for some closed set $S' \subset \hat{X}$.

Let $\mathcal{H}$ be an internal Hilbert space, and $T : \mathcal{H} \to \mathcal{H}$ be an internal bounded operator such that the bound $\|T\|$ is finite (such operator is called *S-bounded*). Define the bounded operator $\hat{T} : \hat{\mathcal{H}} \to \hat{\mathcal{H}}$, called the *standard part* of T, by $\hat{T}\hat{x} = \widehat{Tx}$ (well-defined).

In the following, we assume the polysaturation (Stroyan and Luxemburg, 1976), i.e., our nonstandard model is κ-saturated with $\mathrm{card}(\mathcal{X}) \le \kappa$ where $\mathcal{X}$ is the superstructure that is the domain of the map $\star$.

3. STANDARDIZATIONS OF INTERNAL OPERATORS

This section outlines our recent mathematical results (Yamashita and Ozawa, 2001) on standardizing internal linear operators.

In nonstandard analysis, *standardizations* of internal (or nonstandard) objects have been studied for constructing standard mathematical objects; e.g., an internal measure space is converted into a measure space in the standard sense, called Loeb space (Albeverio *et al.*, 1986). The standardization of an internal Hilbert space $\mathcal{H}$ is the nonstandard hull $\hat{\mathcal{H}}$ of $\mathcal{H}$ (Moore, 1976). The standardization of an internal operator A on $\mathcal{H}$ with finite norm is the standard part $\hat{A}$ of A.

On the other hand, in the case where the norm of A is not finite, it is not straightforward to give an adequate definition of the standard part of A. Albeverio *et al.* (1986) defined $\hat{A}$ only when $\mathcal{H}$ is hyperfinite-dimensional real Hilbert space and A is an internal positive symmetric operator on $\mathcal{H}$.

In this section, we give a definition of $\hat{A}$ for any internal complex Hilbert space $\mathcal{H}$ and for any internal bounded operator A on $\mathcal{H}$, as well as the basic properties on $\hat{A}$ so defined, which suggests the adequacy of the definition. For further information, we refer to Yamashita and Ozawa (2001).

The following proposition enables us to give the first definition of the standard part of A.

Proposition 3.1. *There exists the unique (possibly unbounded) self-adjoint operator S on $\hat{\mathcal{K}}$ satisfying*

$$(S + i)^{-1} = [(A + i)^{-1}]\widehat{\ } \restriction \hat{\mathcal{K}}. \tag{1}$$

Definition 3.1. Under the condition of Proposition 3.1, define the self-adjoint operator $\mathrm{st}_1(A)$ on $\hat{\mathcal{K}}$ by $(\mathrm{st}_1(A)+i)^{-1} = [(A+i)^{-1}]\widehat{\ } \restriction \hat{\mathcal{K}}$.

The operator $\mathrm{st}_1(A)$ is called the *standard part* of A. We see that $\mathrm{st}_1(A) = \hat{A}$ when A is S-bounded.

Definition 3.2. Let A be an internal bounded operator on $\mathcal{H}$, an internal Hilbert space. Define $\mathrm{fin}(A) \subseteq \mathcal{H}$ by

$$\mathrm{fin}(A) = \{\xi \in \mathrm{fin}\mathcal{H} \mid A\xi \in \mathrm{fin}\mathcal{H}\}. \tag{2}$$

Definition 3.3. Let A be an internal bounded self-adjoint operator on $\mathcal{H}$. Let $\hat{\mathcal{K}}$ be the closure of the subspace $[\mathrm{fin}(A)]\widehat{\ } = \{\hat{\xi} \mid \xi \in \mathrm{fin}(A)\}$ of $\hat{\mathcal{H}}$. Define the self-adjoint operator $\mathrm{st}_2(A)$ on $\hat{\mathcal{K}}$ by

$$e^{it\,\mathrm{st}_2(A)} = \widehat{e^{itA}} \restriction \hat{\mathcal{K}} \quad t \in \mathbf{R}. \tag{3}$$

We see that $\{\widehat{e^{itA}} \restriction \hat{\mathcal{K}}\}_{t\in\mathbf{R}}$ is a one-parameter unitary group, since $\hat{\mathcal{K}}$ is invariant under $\widehat{e^{itA}}$ for all $t \in \mathbf{R}$. We also see that it is strongly continuous, as follows. Let $\xi \in \mathrm{fin}(A)$. Then, we have $\|({}^\star d/dt)e^{itA}\xi\| = \|ie^{itA}A\xi\| < \infty$, where ${}^\star d/dt$ is the internal differentiation. This implies that $\widehat{e^{itA}}\hat{\xi}$ is continuous with respect to $t \in \mathbf{R}$. Thus, $\widehat{e^{itA}}$ is strongly continuous on $\mathrm{fin}(A)\widehat{\ }^{\perp\perp}$. Hence by Stone's theorem, $\mathrm{st}_2(A)$ is uniquely defined. If A is S-bounded, $\mathrm{st}_2(A)$ coincides with $\hat{A}$ defined in Section 2.

Let $E(\cdot)$ be an internal projection-valued measure on ${}^\star\mathbf{R}$, i.e., for each internal Borel set $\Omega \subseteq {}^\star\mathbf{R}$, $E(\Omega)$ is an orthogonal projection on $\mathcal{H}$ such that

1. $E(\phi) = 0$, $E({}^\star\mathbf{R}) = I$
2. If $\Omega = \bigcup_{n=1}^{{}^\star\infty} \Omega_n$ with $\Omega_n \cap \Omega_m = \phi$ if $n \neq m$, then $E(\Omega) = \text{s-lim}_{N\to{}^\star\infty} \sum_{n=1}^{N} E(\Omega_n)$
3. $E(\Omega_1)E(\Omega_2) = E(\Omega_1 \cap \Omega_2)$.

For $r \in {}^\star\mathbf{R}$, let $\mathcal{H}_r = \mathrm{Rng}(E(-r,r))$, the range of $E((-r,r))$. Let $D(E) = \bigcup_{r\in\mathbf{R}^+} \mathcal{H}_r \cap \mathrm{fin}\mathcal{H}$. $D(E)$ is called the *standardization domain* of $E(\cdot)$. Clearly, $\widehat{D(E)}^{\perp\perp} = (\bigcup_{r\in\mathbf{R}^+} \hat{\mathcal{H}}_r)^{\perp\perp}$.

For $a \in \mathbf{R}$, define the orthogonal projection $\hat{E}_{\mathrm{st}}(-\infty, a]$ by

$$\hat{E}_{\mathrm{st}}(-\infty, a] = \sup\{\hat{E}(-K, a+\epsilon] \restriction \hat{D}(E)^{\perp\perp} \mid K, \epsilon \in \mathbf{R}^+\} \tag{4}$$

$$= \operatorname*{s-lim}_{n\to\infty} \hat{E}\left(-n, a+\frac{1}{n}\right] \restriction \hat{D}(E)^{\perp\perp}. \tag{5}$$

Then we see

$$\operatorname*{s-lim}_{a\to-\infty} \hat{E}_{\mathrm{st}}(-\infty, a] = 0 \tag{6}$$

$$\operatorname*{s-lim}_{\epsilon\downarrow 0} \hat{E}_{\rm st}(-\infty, a+\epsilon] = \hat{E}_{\rm st}(-\infty, a] \tag{7}$$

$$a < b \Rightarrow \hat{E}_{\rm st}(-\infty, a] \leq \hat{E}_{\rm st}(-\infty, b]\,. \tag{8}$$

Hence, $\hat{E}_{\rm st}(-\infty, \cdot]$ defines a projection-valued measure on $\mathbf{R}$.

Definition 3.4. For any internal bounded self-adjoint operator A, define the self-adjoint operator $\mathrm{st}_3(A)$ on $\hat{D}(E)^{\perp\perp}$ by

$$\mathrm{st}_3(A) = \int \lambda d\hat{E}_{\rm st}(\lambda). \tag{9}$$

Proposition 3.2. *Let A be an internal bounded self-adjoint operator, and $E(\cdot)$ the internal projection-valued measure associated with the spectral decomposition of A. Then*

$$\hat{D}(E)^{\perp\perp} = \widehat{\mathrm{fin}(A)}^{\perp\perp}. \tag{10}$$

Theorem 3.3. *Definition 3.1, 3.3, and 3.4 are equivalent, i.e.,* $\mathrm{st}_1(A) = \mathrm{st}_2(A) = \mathrm{st}_3(A)$.

In Section 2, $\hat{A}$ is defined only when A is an internal S-bounded self-adjoint operator. Now we can extend the definition so as to include the case where A is an internal bounded self-adjoint operator that is not S-bounded; $\hat{A} := \mathrm{st}_1(A) = \mathrm{st}_2(A) = \mathrm{st}_3(A)$.

Definition 3.5. Let A be an internal linear operator on an internal Hilbert space $\mathcal{H}$. Let D be an (external) subspace of $\mathrm{fin}\mathcal{H}$. A is *standardizable* on D if $D \subset \mathrm{fin}(A)$ and if for any $x, y \in D$, $x \approx y$ implies $Ax \approx Ay$. In this case, define the operator $\hat{A}_D$ with domain $\hat{D} = \{\hat{x} \mid x \in D\}$, called the *standard part* of A on D, by

$$\hat{A}_D\hat{x} = \widehat{Ax}, \quad x \in D. \tag{11}$$

Clearly, A is standardizable on D if and only if $D \subset \mathrm{fin}(A)$, and if $A\xi \approx 0$ for all $\xi \in D$ with $\xi \approx 0$.

Lemma 3.4. *An internal bounded operator A is standardizable on* $\mathrm{fin}(A^*A)$.

Corollary 3.5. *If $D \subseteq \mathrm{fin}\mathcal{H}$ is invariant under A and A^*, A is standardizable on D.*

Theorem 3.6. *Let A be an internal self-adjoint operator on $\mathcal{H}$, and $E(\cdot)$ the projector-valued spectral measure of A. Then,*

$$\hat{A} = \overline{\hat{A}_{D(E)}} = \overline{\hat{A}_{\mathrm{fin}(A^2)}} \tag{12}$$

For a usual (resp. internal) closed operator A, let $\sigma(A)$ and $\sigma_p(A)$ denote the usual (resp. internal) spectrum and the usual (resp. internal) point spectrum of A, respectively.

Theorem 3.7. *For an internal self-adjoint operator A,*

$$\sigma(\hat{A}) = \sigma_p(\hat{A}) = \mathrm{st}[\sigma(A) \cap \mathrm{fin}\ {}^{\star}\mathbf{R}]. \tag{13}$$

Definition 3.6. Let A be an internal bounded self-adjoint operator on $\mathcal{H}$. Let H be a subspace of fin$\mathcal{H}$. A real number λ is a *approximate eigenvalue* of *A relative to H*, if for any internal set $S \supset H$, there is $\xi \in S$ with $\|\xi\| = 1$ such that $(A - \lambda)\xi \approx 0$.

We denote by $\sigma_{\mathrm{app}}(A; H)$ the set of approximate eigenvectors of A relative to H.

Let st the standardizing operation on $\mathcal{H}$, i.e., $\mathrm{st}(\xi) = \hat{\xi}$. We easily see $H \subset \mathrm{st}^{-1}(\hat{H})$.

Theorem 3.8. $\lambda \in \sigma_{\mathrm{app}}(A; H)$ *if and only if there exists a sequence* $\{\xi_i\}_{i \in N} \subset H$ *with* $\|\xi_i\| = 1$ *such that* $\lim_{i\to\infty} {}^{\circ}\|(A - \lambda)\xi_i\| = 0$.

Corollary 3.9. *Let A be an internal bounded self-adjoint operator. Let $H \subset \mathrm{fin}\mathcal{H}$ be a subspace such that $\hat{H}$ is closed and $H = \mathrm{st}^{-1}(\hat{H})$. Denote by $\hat{A}(\hat{H})$ the self-adjoint operator on $\hat{H}^{\perp\perp}$ defined by $\hat{A}(\hat{H}) = \hat{A} \mid \hat{H} \cap \mathrm{dom}(\hat{A})$. Then,*

$$\sigma(\hat{A}(\hat{H})) = \sigma_{\mathrm{app}}(A; H). \tag{14}$$

Definition 3.7. Let A and B be usual possibly unbounded self-adjoint operators on a (usual) Hilbert space $\mathcal{H}$. The operators A and B *commute* if e^{isA} and e^{itB} commute for each real t.

It is known that A and B commute iff any spectral projection of A commute with any spectral projection of B.

Definition 3.8. Let A and B be internal bounded self-adjoint operators on $\mathcal{H}$, and H a subspace of fin$\mathcal{H}$. The operators A and B are *approximately commute* on H if

$$e^{isA}\, e^{itB}\xi \approx e^{itB}\, e^{itA}\xi \tag{15}$$

for all $\xi \in H$ and $s, t \in \mathbf{R}$.

Definition 3.9. Let $A_1, \ldots, A_n$ ($n \in$ N) be internal bounded self-adjoint operators, and H a subspace of fin$\mathcal{H}$. Define $\sigma_{\mathrm{app}}(A_1, \ldots, A_n; H) \subseteq \mathbf{R}$ to be the set of $\lambda \in \mathbf{R}$ such that for any internal set $S \supset H$, there is $\xi \in S$ such that $\|\xi\| = 1$ and $A_i\xi \approx \lambda_i\xi$ $(i = 1, \ldots, n)$. Each element of $\sigma_{\text{app}}(A_1, \ldots, A_n; H)$ is called an *approximate joint eigenvalue* of $A_1, \ldots, A_n$ on H.

Theorem 3.10. *Let $A_1, \ldots, A_n$ $(n \in \mathbf{N})$ be internal bounded self-adjoint operators on $\mathcal{H}$, approximately commuting on $H \subset \text{fin}\mathcal{H}$. Suppose that H is S-closed and $\hat{H}$ is invariant under each $\hat{A}_i$. Then, the self-adjoint operators $\hat{A}_1 \restriction \hat{H} \cap \text{dom}(\hat{A}_1), \ldots, \hat{A}_n \restriction \hat{H} \cap \text{dom}(\hat{A}_n)$ commute, and $\sigma_{\text{app}}(A_1, \ldots, A_n; H)$ coincides with their joint spectrum.*

4. NONSTANDARD REPRESENTATIONS OF THE CCR

Definition 4.1. Let $\mathcal{H}$ be an internal Hilbert space, and $\mathcal{K}$ be an internal inner product space. Let $a(\cdot)$ be an internal map from $\mathcal{K}$ to the internal bounded operators on $\mathcal{H}$. Let $D_{\mathcal{H}}$ and $D_{\mathcal{K}}$ be subspaces of $\text{fin}\mathcal{H}$ and $\text{fin}\mathcal{K}$ respectively. The triple $(a(\cdot), D_{\mathcal{H}}, D_{\mathcal{K}})$ is called a (nonstandard) representation of the CCR if

(i) $f \mapsto a(f)^*$ is linear.
(ii) $[a(f), a(g)^*]\xi \approx \langle f, g\rangle\xi$, $[a(f), a(g)] = 0$ $f, g \in D_{\mathcal{K}}, \xi \in D_{\mathcal{H}}$
(iii) $D_{\mathcal{H}}$ is invariant under $a(f)$ and $a(f)^*$ for all $f \in D_{\mathcal{K}}$.

Definition 4.2. Let $(a(\cdot), D_{\mathcal{H}}, D_{\mathcal{K}})$ be a representation of the CCR such that if $f, g \in D_{\mathcal{K}}$ and $\|f - g\| \approx 0$ then $a(f)\xi \approx a(g)\xi$, $a(f)^*\xi \approx a(g)^*\xi$ for all $\xi \in D_{\mathcal{H}}$. The representation $(a(\cdot), D_{\mathcal{H}}, D_{\mathcal{K}})$ is called *S-continuous*. Define the maps $\hat{a}(\cdot)$ and $\hat{a}^\dagger(\cdot)$ from $\hat{D}_{\mathcal{K}} = \{\hat{f} \mid f \in D_{\mathcal{K}}\}$ to the operators on the inner product space $\hat{D}_{\mathcal{H}} = \{\hat{\xi} \mid \xi \in D_{\mathcal{H}}\}$ by

$$\hat{a}(\hat{f})\hat{\xi} = (a(f)\xi)\widehat{\ }, \qquad \hat{a}^\dagger(\hat{f})\hat{\xi} = (a(f^*)\xi)\widehat{\ }, \quad \xi \in D_{\mathcal{H}}, \ f \in D_{\mathcal{K}}. \tag{16}$$

We see that these are well-defined by the following: Condition (iii) implies $D_{\mathcal{H}} \subseteq \text{fin}(a(f)^*a(f)) \cap \text{fin}(a(f)a(f)^*)$ for all $f \in D_{\mathcal{K}}$. Thus by Lemma 3.4, $a(f)$ and $a(f)^*$ are standardizable on $D_{\mathcal{H}}$, i.e., if $\xi, \zeta \in D_{\mathcal{H}}$ and $\xi \approx \zeta$ then $a(f)\xi \approx a(f)\zeta$ and $a(f)^*\xi \approx a(f)^*\zeta$. The operators $\hat{a}(\cdot)$ and $\hat{a}^\dagger(\cdot)$ give a standard representation of the CCR:

$$[\hat{a}(\hat{f}), \hat{a}(\hat{g})] = 0, \qquad [\hat{a}(\hat{f}), \hat{a}^\dagger(\hat{g})] = \langle \hat{f}, \hat{g}\rangle \equiv {}^\circ\langle f, g\rangle \tag{17}$$

Consider the case where $\mathcal{K} = {}^\star\mathbf{C}$, $D_{\mathcal{K}} = \text{fin}^\star\mathbf{C}$ and $\mathcal{H}$ is an hyperfinite-dimensional internal Hilbert space.

Example 1 (number truncation). Let H be a (standard) separable Hilbert space. Let A and $A^\dagger$ be the annihilation and the creation operators on H: let $\{e_i\}_{i\in\mathbf{N}}$ denote a complete orthonormal system of H, and $D \subset H$ be spanned by $\{e_i\}_i$. The operators A and $A^\dagger$ are defined by

$$Ae_0 = 0, \qquad Ae_i = \sqrt{i}e_{i-1}, \qquad A^\dagger e_i = \sqrt{i+1}e_{i+1}. \tag{18}$$

Let $\{\psi_i\}_{i\in {}^\star\mathbf{N}} = {}^\star(\{e_i\}_{i\in\mathbf{N}})$ where $\psi_i = {}^\star e_i$ for $i \in \mathbf{N}$, and $\mathcal{H}$ be the internal Hilbert space spanned by $\psi_0, \ldots, \psi_\nu$ ($\nu \in {}^\star\mathbf{N}_\infty$). Define the internal operators a and a^* on $\mathcal{H}$ by projecting the operators A and A^*:

$$a\psi_0 = 0, \qquad a\psi_i = \sqrt{i}\epsilon_{i-1}, \quad i = 1, \ldots, \nu, \tag{19}$$

$$a^*\psi_i = \sqrt{i+1}\epsilon_{i+1}, \quad i = 1, \ldots, i-1, \tag{20}$$

$$a^*\psi_\nu = 0. \tag{21}$$

The operators a and a^* are called the *number-truncated annihilation operator* and *number-truncated creation operator*.

Let $a(z) = \overline{z}a$ for $z \in \mathcal{K} = {}^\star\mathbf{C}$. Let $D_\mathcal{H} = {}^\sigma D$ and $D_\mathcal{K} = \mathrm{fin}^\star\mathbf{C}$. We see $(a(\cdot), D_\mathcal{H}, D_\mathcal{K})$ is a S-continuous representation of the CCR. The *number operator* N satisfying

$$e^{itN}a\, e^{-itN} = e^{-it}a, \quad t \in {}^\star\mathbf{R}, \tag{22}$$

and

$$N\xi = 0 \quad \text{if and only if} \quad a\xi = 0 \tag{23}$$

is given by $N = a^*a$.

Example 2 (hyperfinite extension). Let H and A be same as Example 1. Let F be an internal hyperfinite-dimensional linear space with ${}^\sigma H \subset F \subset {}^\star H$, and the operator a on F be a hyperfinite extension of A. Set $D_\mathcal{H} = {}^\sigma D$ and $D_\mathcal{K} = \mathrm{fin}^\star\mathbf{C}$. Define $a(\cdot)$ by $a(z) = \overline{z}a$ ($z \in \mathrm{fin}^\star\mathbf{C}$). Then, $(a(\cdot), D_\mathcal{H}, D_\mathcal{K})$ is an S-continuous representation of the CCR.

Example 3 (spin matrices). Let J_1, J_2, and J_3 be the hermitian generators of ν-dimensional internal irreducible representation of su(2) with the representation space $\mathcal{H}$, satisfying

$$[J_k, J_l] = i\epsilon_{klm}J_m. \tag{24}$$

Let $a = (J_1 + J_2)/\sqrt{j}$ with $j = (\nu - 1)/2$, and $a(z) = \overline{z}a$ for $z \in \mathcal{K} = {}^\star\mathbf{C}$. Let $\Omega \in \mathcal{H}$ satisfy $\|\Omega\| = 1$ and $a\Omega = 0$. Let $D_\mathcal{H}$ be spanned by Ω and $a_k^*\Omega$ ($k \in \mathbf{N}$), and $D_\mathcal{K} = \mathrm{fin}^\star\mathbf{C}$. Then, $(a(\cdot), D_\mathcal{H}, D_\mathcal{K})$ is a S-continuous representation of the CCR. The number operator N satisfying (22) and (23) is given by $N = J_3 + j$. Note that $N \neq a^*a$ in this example. Instead we have $N = (j/2)[a, a^*] + j$ and $a^*a = N(1 - N/\nu)$.

Now we will consider a case where $\mathcal{K}$ is infinite-dimensional.

Example 4 (number truncation for the case of infinite degree of freedom). Let J be a hyperfinite set with $|J| \in {}^\star\mathbf{N}_\infty$. Let $\mathcal{K} = {}^\star\mathbf{C}^J$ with the inner product $\langle f, g \rangle = \epsilon \sum_{i \in J} \overline{f(i)} g(i)$ ($f, g \in {}^\star\mathbf{C}^J$), where $\epsilon \in {}^\star\mathbf{R}$ is a fixed constant. Let a be the number-truncated annihilation operator on $\mathcal{H}_0$. Let $\mathcal{H} = \otimes_{j \in J} \mathcal{H}_0$ and $a_i = \otimes_{j \in J} a_{ij}$, where $a_{ij} = a$ if $i = j$, and $a_{ij} = 1$ otherwise. For $f \in {}^\star\mathbf{C}^J$, define $a(f)$ by

$$a(f) = \sqrt{\epsilon} \sum_{i \in J} \overline{f(i)} a_i. \tag{25}$$

Let $\mathcal{A}$ be the (external) *-algebra generated by $\{1, a(f) \mid f \in \text{fin}^\star\mathbf{C}^J\}$, and set $\Omega = \otimes_{j \in J} \psi_0$, $D_\mathcal{H} = \mathcal{A}\Omega$ and $D_\mathcal{K} = \text{fin}^\star\mathbf{C}^J$. Then, we find that $(a(\cdot), D_\mathcal{H}, D_\mathcal{K})$ is a representation of the CCR. Condition (i) of Definition 4.1 is clear. Condition (ii) is shown as follows. Note that $[a_i, a_j^*] a_{i_1}^* \cdots a_{i_n}^* \Omega = \delta_{ij} a_{i_1}^* \cdots a_{i_n}^* \Omega$ ($n \in \mathbf{N}$). Hence, we have e.g., $a_{i_1} a_{i_2}^* a_{i_3}^* \Omega = (\delta_{i_1 i_3} + \delta_{i_1 i_2} a_{i_3}^*)\Omega$. Thus, every vector in $D_\mathcal{H} = \mathcal{A}\Omega$ is an internal linear combination of the vectors of the form $a_{i_1}^* \cdots a_{i_n}^* \Omega$. Thus, $[a_i, a_j^*]\xi = \delta_{ij}\xi$ for all $\xi \in D_\mathcal{H}$, and hence $[a(f), a(g)^*]\xi = \epsilon \sum_{i,j \in J} \overline{f(i)} g(j)$ $[a_i, a_j^*]\xi = \sum_{i \in J} \overline{f(i)} g(i)\xi = \langle f, g \rangle \xi$. Therefore (ii) is satisfied. Condition (iii) is shown as follows. Clearly, $D_\mathcal{H}$ is invariant under $\mathcal{A}$. Let $a(f)^\sharp$ denote $a(f)$ or $a(f)^*$. By a straightforward calculation, we can check $\langle \Omega, a(f_1)^\sharp \cdots a(f_1)^\sharp \Omega \rangle$ ($f_1, \ldots, f_n \in \text{fin}^\star\mathbf{C}^J$, $n \in \mathbf{N}$) is finite. Hence $D_\mathcal{H} \subseteq \text{fin}\mathcal{H}$. Thus (iii) is satisfied.

The representation $(a(\cdot), D_\mathcal{H}, D_\mathcal{K})$ also is S-continuous. In fact, if $\|f\| \approx 0$, $\|f\| \neq 0$ and $\xi \in D_\mathcal{H}$, then $a(f)^\sharp \xi = \|f\| a(f/\|f\|)^\sharp \xi \approx 0$, becasue $f/\|f\| \in D_\mathcal{K}$ and $a(f/\|f\|)^\sharp \xi \in D_\mathcal{H} \subseteq \text{fin}\mathcal{H}$. So we can consider the standardization of the representation as stated after Definition 4.2: for any vector $\hat{\xi}$ in the nonseparable Hilbert space $\hat{D}_\mathcal{K} = ({}^\star\mathbf{C}^J)\widehat{\ }$, $\hat{a}(\hat{f})$ and $\hat{a}^\dagger(\hat{f})$ becomes standard operators defined on $\hat{D}_\mathcal{H}$. Let $\hat{\mathcal{A}}$ be the algebra generated by $\{1, \hat{a}(\hat{f}), \hat{a}^\dagger(\hat{f}) \mid \hat{f} \in \hat{D}_\mathcal{K}\}$. We see $\hat{D}_\mathcal{H} = \hat{\mathcal{A}}\hat{\Omega}$, i.e., this representation is cyclic.

5. HYPERFINITE WIGHTMAN AXIOMS

Let $\mathcal{S}(\mathbf{R}^l)$ ($l \in \mathbf{N}$) denote the space of the functions of rapid decrease with the usual topology. Since $\mathcal{S}(\mathbf{R}^l)$ is Hausdorff, we can define the equivalence relation $\stackrel{\mathcal{S}}{\approx}$ by

$$f \stackrel{\mathcal{S}}{\approx} g \quad \text{if and only if} \quad f - g \in \text{mon}_\mathcal{S}(0). \tag{26}$$

Definition 5.1. The triple (φ, Ω, P) of an internal linear map φ from ${}^\star\mathcal{S}(\mathbf{R}^l)$ to the internal operators on an internal hyperfinite-dimensional Hilbert space F, and a unit vector $\Omega \in F$, and internal self-adjoint operators $P = (P_0, \ldots, P_{l-1})$, is called a hyperfinite Hermitian scalar field theory if it satisfies the following properties:

Let $\mathcal{A}$ be the algebra generated by $\{I, \varphi(f) \mid f \in {}^\sigma\mathcal{S}(\mathbf{R}^l)\}$.

(HW1) $\mathcal{A}\Omega \subseteq \text{fin}F$.
(HW2) If f is real-valued then $\varphi(f)$ is self-adjoint.
(HW3) If $F_1, F_2 \in \text{ns}^\star\mathcal{S}(\mathbf{R}^l)$ and $F_1 \overset{\mathcal{S}}{\approx} F_2$, then

$$\langle \xi_1, \varphi(F_1)\xi_2\rangle \approx \langle \xi_1, \varphi(F_2)\xi_2\rangle \quad \xi_1, \xi_2 \in \mathcal{A}\Omega. \tag{27}$$

(HW4) For $f_1, \ldots, f_n \in {}^\sigma\mathcal{S}(\mathbf{R}^l)$ and $(a, \Lambda) \in P_+^\uparrow$, the restricted Poincaré group,

$$\langle \Omega, \varphi(f_1)\cdots\varphi(f_n)\Omega\rangle \approx \langle \Omega, \varphi(f_{1(a,\Lambda)})\cdots\varphi(f_{n(a,\Lambda)})\Omega\rangle, \tag{28}$$

where $f_{(a,\Lambda)}(x) = f(\Lambda^{-1}(x-a))$.
(HW5)
  (a) $\widehat{\mathcal{A}\Omega}$ is invariant under $(e^{-ia^\mu P_\mu})\widehat{}$ for all $a \in \mathbf{R}^l$.
  (b) For any $a \in \mathbf{R}^l$, $e^{-ia^\mu P_\mu}\Omega \approx \Omega$.
  (c) $(e^{ia^\mu P_\mu})\widehat{}\,\hat{\varphi}_{\mathcal{A}\Omega}(f)(e^{-ia^\mu P_\mu})\widehat{}\,\hat{\xi} = \hat{\varphi}_{\mathcal{A}\Omega}(f_{(a,1)})\hat{\xi},\ \xi \in \mathcal{A}\Omega$.
(HW6) $P_0, \ldots, P_{l-1}$ approximately commute on $\mathcal{A}\Omega$, and $\sigma_{\text{app}}(P; \widetilde{\mathcal{A}\Omega})$ is a subset of the closed forward light cone.
(HW7) If the supports of f and g are space-like separated,

$$[\varphi(f), \varphi(g)]\xi \approx 0 \quad \xi \in \mathcal{A}\Omega,\ f, g \in {}^\sigma\mathcal{S}(\mathbf{R}^l). \tag{29}$$

(HW8) If $e^{-ia^\mu P_\mu}\xi \approx \xi$ for all $a \in \mathbf{R}^l$, $\xi \approx c\Omega\, (c \in \mathbf{C})$.

Let us call HW1–HW8 the *hyperfinite Wightman axioms*.

Definition 5.2 (Gårding and Wightman, 1964; Streater and Wightman, 1964). The quadruple $(\mathcal{H}, \Omega, \phi, U)$ of a separable Hilbert space $\mathcal{H}$, a map ϕ from $\mathcal{S}(\mathbf{R}^l)$ to the operators on $\mathcal{H}$, and a strongly continuous representation U of $P_+^\uparrow$ on $\mathcal{H}$, is called a *Gårding–Wightman Hermitian scalar field theory* if it satisfies the following properties:

(GW1) There exists a dense domain $D_0 \subset \mathcal{H}$ satisfying the following (a)–(d):
  (a) For each $f \in \mathcal{S}(\mathbf{R}^l)$, $\text{dom}(\phi(f)) \subseteq D_0$, and for all $\xi_1, \xi_2 \in D_0$, the linear functional $\langle \xi_1, \phi(\cdot)\xi_2\rangle$ is a tempered distribution.
  (b) If $f \in \mathcal{S}(\mathbf{R}^l)$ is real-valued, $\phi(f)$ is symmetric on D_0.
  (c) For each $f \in \mathcal{S}(\mathbf{R}^l)$, D_0 is invariant under $\phi(f)$.
  (d) $\Omega \in D_0$ and D_0 is algebraically spanned by $\{\Omega, \phi(f_1)\cdots\phi(f_n)\Omega \mid f_1, \ldots, f_n \in \mathcal{S}(\mathbf{R}^l), n \in \mathbf{N}\}$.
(GW2)
  (a) For any $(a, \Lambda) \in P_+^\uparrow$, D_0 is invariant under $U(a, \Lambda)$.
  (b) For any $(a, \Lambda) \in P_+^\uparrow$, $U(a, \Lambda)\Omega = \Omega$.
  (c) For any $f \in \mathcal{S}(\mathbf{R}^l)$ and $(a, \Lambda) \in P_+^\uparrow$,

$$U(a, \Lambda)\phi(f)U(a, \Lambda)^{-1} = \phi(f_{(a,\Lambda)}). \tag{30}$$

(GW3) The joint spectrum of the infinitesimal generators $\mathbf{P}_0, \ldots, \mathbf{P}_{l-1}$ of the subgroup $\{U(a,1) \mid a \in \mathbf{R}^l\}$ is a subset of the closed forward cone.
(GW4) If the supports of f and g are space-like separated, $[\phi(f), \phi(g)] = 0$ on D_0.
(GW5) A vector invariant under each $U(a,1)$ $(a \in \mathbf{R}^l)$ is a scalar multiple of Ω.

We will show the equivalence of the Gårding–Wightman axioms and the hyperfinite Wightman axioms.

Let $(\mathcal{H}, \phi, \Omega, U)$ be a Gårding–Wightman Hermitian scalar field theory. Let F be an internal hyperfinite-dimensional Hilbert space such that ${}^\sigma\mathcal{H} \subset F \subset {}^\star\mathcal{H}$, and E the orthogonal projection onto F. Then, $\varphi(\cdot) = E{}^\star\phi(\cdot) \restriction F$ is an internal linear map from ${}^\star\mathcal{S}(\mathbf{R}^l)$ to the internal operators on F. Let $\mathcal{A}$ be the *-algebra generated by $\{1, \varphi({}^\star f) \mid f \in \mathcal{S}(\mathbf{R}^l)\}$. Since $\mathcal{A}\Omega = {}^\sigma D_0$ by GW1(d), HW1 holds. If $f \in \mathcal{S}(\mathbf{R}^l)$ is real-valued, $\varphi({}^\star f)$ is self-adjoint HW2. HW3 follows from GW1(a). HW4 follows from GW2.

To show HW5, we need the following lemmas:

Lemma 5.1. *Let $m, n, l \in \mathbf{N}$. For any bounded operators $T_1, \ldots, T_m$, and positive real ϵ, any vectors $\psi_1, \ldots, \psi_n \in \mathcal{H}$ and any real $t_1, \ldots, t_l$, there is an orthogonal projection E of finite rank such that*

$$\|e^{it_\alpha ET_\beta E}\psi_\gamma - e^{it_\alpha T_\beta}\psi_\gamma\| < \epsilon \quad \text{and} \quad E\psi_\gamma = \psi_\gamma \tag{31}$$

for $\alpha = 1, \ldots, l$, $\beta = 1, \ldots, m$, and $\gamma = 1, \ldots, n$.

Proof: Let $N \in \mathbf{N}$ and E_N be the projector onto the finite-dimensional subspace spanned by $\{T_\beta^k \psi_\gamma \mid k = 0, \ldots, N,\ \beta = 1, \ldots, m,\ \gamma = 1, \ldots, n\}$. We have

$$\begin{aligned} d_{\alpha\beta\gamma}^{(N)} &:= \|e^{it_\alpha E_N T_\beta E_N}\psi_\gamma - e^{it_\alpha T_\beta}\psi_\gamma\| \\ &= \left\| \sum_{k=N+1}^{\infty} \frac{(it_\alpha E_N T_\beta E_N)^k}{k!}\psi_\gamma - \sum_{k=N+1}^{\infty} \frac{(it_\alpha T_\beta)^k}{k!}\psi_\gamma \right\| \to 0 \quad (N \to \infty). \end{aligned}$$

Thus, take a sufficiently large N so that $\max_{\alpha,\beta,\gamma} d_{\alpha\beta\gamma}^{(N)} < \epsilon$, and let $E = E_N$. □

Lemma 5.2. *Let $\mathcal{H}$ be a Hilbert space, and T_k $(k \in \mathbf{N})$ internal bounded operators on ${}^\star\mathcal{H}$. There exists a hyperfinite-dimensional internal subspace F with ${}^\sigma\mathcal{H} \subset F \subset {}^\star\mathcal{H}$ such that for every $k \in \mathbf{N}$, every real t, and every vector $\xi \in {}^\sigma\mathcal{H}$,*

$$e^{itET_kE}\xi \approx e^{itT_k}\xi, \tag{32}$$

where E is the internal orthogonal projection onto F.

Proof: By the Transfer Principle, Lemma 5.1 is transferred to: “for any internal bounded operators $T_1, \ldots, T_m$, any positive hyperreal ϵ, any vectors

$\psi_1, \ldots, \psi_n \in {}^\star\mathcal{H}$ and any hyperreals $t_1, \ldots, t_l$, there is an internal orthogonal projection P of hyperfinite rank such that (31) holds." Thus we find that the internal relation

$$\|e^{itPT_kP}\psi - e^{itT_k}\psi\| < \epsilon \quad \text{and} \quad P\psi = \psi \tag{33}$$

with the two variables (k, ψ, t, ϵ) and P is concurrent (Hurd and Loeb, 1985; Stroyan and Luxemburg, 1976) on $\mathbf{N} \times {}^\sigma\mathcal{H} \times \mathbf{R} \times \mathbf{R}^+$. Thus, by the saturation, the proof is completed. □

Lemma 5.3. *Let $A_1, \ldots, A_n$ $(n \in \mathbf{N})$ be (possibly unbounded) commuting self-adjoint operators on $\mathcal{H}$, a Hilbert space. Then, there are internal bounded commuting self-adjoint operators $B_1, \ldots, B_n$ on ${}^\star\mathcal{H}$ such that*

$$e^{it^\star A_i}{}^\star\xi \approx e^{itB_i}{}^\star\xi, \quad i = 1, \ldots, n, \tag{34}$$

for any $\xi \in \mathcal{H}$ and $t \in \mathbf{R}$.

Proof: Let $P(\cdot)$ be the internal projection-valued joint-spectral measure on ${}^\star\mathbf{R}^n$ associated with $A_1, \ldots, A_n$. Let S be an internal bounded Borel set such that $\text{fin}\,{}^\star\mathbf{R}^n \subset S \subset {}^\star\mathbf{R}^n$, and set $B_i = P(S)A_iP(S)$. Since $P(S)^\star\xi \approx {}^\star\xi$ for any $\xi \in \mathcal{H}$, (34) holds. □

We apply Lemma 5.3 to the self-adjoint operators $\mathbf{P}_0, \ldots, \mathbf{P}_{l-1}$ on $\mathcal{H}$, and have internal commuting bounded self-adjoint operators $\mathbf{P}'_0, \ldots, \mathbf{P}'_3$, satisfying $e^{it\mathbf{P}'_i}{}^\star\xi \approx e^{it^\star\mathbf{P}_i}{}^\star\xi$ $(i = 1, \ldots, 3)$ for any $t \in \mathbf{R}$ and $\xi \in \mathcal{H}$. By Lemma 5.2, we can assume $e^{itE\mathbf{P}'_iE}{}^\star\xi \approx e^{it\mathbf{P}'_i}{}^\star\xi$ for any $t \in \mathbf{R}$ and $\xi \in \mathcal{H}$. Set $\mathbf{P}_i = E\mathbf{P}'_i \restriction \mathcal{K}$, and we have

$$e^{itP_i}{}^\star\xi \approx e^{it^\star\mathbf{P}_i}{}^\star\xi, \quad \xi \in \mathcal{H}, i = 0, \ldots, l-1. \tag{35}$$

Thus, HW5(a)–(c) follows from GW2(a)–(c). The commutability of $\mathbf{P}_0, \ldots, \mathbf{P}_3$ and (35) imply that the approximate commutability of $P_1, \ldots, P_{l-1}$ on ${}^\sigma\mathcal{H}$. HW6 is shown by the following:

Theorem 5.4. $\hat{P}_\alpha \restriction [{}^\sigma\text{dom}(\mathbf{P}_\alpha)]\widehat{\ }$ *is unitarily equivalent to* $\mathbf{P}_\alpha$ $(\alpha = 0, \ldots, 3)$.

Proof: Since $\mathbf{P}_\alpha + i$ is a bijection from $\text{dom}(\mathbf{P}_\alpha)$ to $\mathcal{H}$, the operator $[({}^\star\mathbf{P}_\alpha + i)^{-1}]\widehat{\ } \restriction \widehat{{}^\sigma\mathcal{H}}$ is a bijection to $[{}^\sigma\text{dom}(\mathbf{P}_\alpha)]\widehat{\ }$. Suppose $\xi, \eta \in {}^\sigma\text{dom}(\mathbf{P}_\alpha)$ and $P_\alpha\xi \approx {}^\star\mathbf{P}_\alpha\eta$. Then, ${}^\star\mathbf{P}_\alpha\xi = E\ {}^\star\mathbf{P}_\alpha\xi \approx E\mathbf{P}'_\alpha\xi = P_\alpha\xi \approx {}^\star\mathbf{P}_\alpha\eta$ because for all $\zeta \in {}^\sigma\mathcal{H}$, $\mathbf{P}_\alpha\zeta \approx {}^\star p\zeta$. Thus we have $\xi \approx \eta$ and hence $\xi = \eta$. It follows that $[({}^\star\mathbf{P}_\alpha + i)^{-1}]\widehat{\ } \restriction \widehat{{}^\sigma\mathcal{H}} = (P_\alpha + i)^{-1}\widehat{\ } \restriction \widehat{{}^\sigma\mathcal{H}} = (\hat{P}_\alpha + i)^{-1} \restriction \widehat{{}^\sigma\mathcal{H}}$. Thus, ${}^\star\mathbf{P}_\alpha\xi = \eta$ if only if $\hat{P}\hat{\xi} = \hat{\eta}$ for $\xi \in {}^\sigma\text{dom}(p)$ and $\eta \in {}^\sigma\mathcal{H}$. Therefore, the unitary map $\psi \mapsto \widehat{{}^\star\psi}$ $(\psi \in \mathcal{H})$ transforms $\mathbf{P}_\alpha$ to $\hat{P}_\alpha \restriction [{}^\sigma\text{dom}(\mathbf{P}_\alpha)]\widehat{\ }$. □

Theorem 5.5. *$\sigma_{\text{app}}(P_0, \ldots, P_{l-1}; \widetilde{{}^{\sigma}\mathcal{H}})$ coincides with the joint spectrum of* $\mathbf{P}_0, \ldots, \mathbf{P}_{l-1}$.

Proof: By Theorem 3.10. □

Theorem 5.6. *For any Gårding–Wightman Hermitian scalar field theory ($\mathcal{H}, \phi, \Omega, U$), there exists a hyperfinite Hermitian scalar field theory (φ, Ω', P) such that the standard part $\widehat{\varphi({}^\star f)}_{\mathcal{A}\Omega}$ ($f \in \mathcal{S}(\mathbf{R}^l)$) is unitarily equivalent to $\phi(f)$.*

Conversely, we can get a Hermitian scalar field theory from a given hyperfinite scalar field theory. Let (φ, Ω, P) be a hyperfinite Hermitian scalar field theory. Let $\mathcal{H} = \widehat{(\mathcal{A}\Omega)}^{\perp\perp}$. By HW1 and Corollary 3.5, we can define a map ϕ from $\mathcal{S}(\mathbf{R}^l)$ to the symmetric operators on $\mathcal{K}$ by $\phi(f) = \widehat{\varphi({}^\star f)}_{\mathcal{A}\Omega}$. Then GW1(a) follows from HW3. The Lorenz invariance of the Wightman functions follows from HW4. Hence, a strongly continuous unitary representation U of the $P_+^\uparrow$ on $\mathcal{H}$ satisfying GW2 is constructed in a similar way as that in the construction occurring in the proof of the Wightman reconstruction theorem (Bogolubov *et al.*, 1975; Streater and Wightman, 1964). Since $U(a, 1) = (e^{ia^\mu \mathbf{P}_\mu}) \restriction \mathcal{H} = e^{ia^\mu \hat{P}_\mu} \restriction \mathcal{H}$, the generators of translations are $\hat{P}'_\alpha = \hat{P}_\alpha \restriction \mathcal{H} \cap \text{dom}(\hat{P}_\alpha)$. By Theorem 3.10, the joint spectrum of $\hat{P}_0, \ldots, \hat{P}_{l-1}$ is a subset of the closed forward light cone.

Theorem 5.7. *For any hyperfinite Hermitian scalar field theory, its standardization defined above gives a Gårding–Wightman Hermitian scalar field theory.*

Any mathematically nontrivial Gårding–Wightman quantum field, where $[\phi(f), \phi(g)] \neq 0$ for some f and g, cannot be represented as $\phi(f) = \int \phi(x) f(x)dx$ with pointwise-defined operator $\phi(x)$. (Note that the meaning of "mathematically nontrivial field" is weaker than the ordinary use of "nontrivial field.") However, in the nonstandard field theory, $\varphi(x)$ can be a well-defined operator; in fact, we shall see in the following example, that there exists a hyperfinite Hermitian scalar field theory (φ, Ω, P) whose standardization is the standard Klein–Gordon field theory that satisfies the Gårding–Wightman axioms, and there exists an internal field operator $\varphi(x)$ well-defined for each nonstandard space–time point $x \in {}^\star\mathbf{R}^l$ such that

$$\varphi(f) = {}^\star\!\int f(x)\varphi(x)\,dx, \quad f \in {}^\star\mathcal{S}(\mathbf{R}^l). \tag{36}$$

It is expected that for any Gårding–Wightman field theory there is a nonstandard field operator φ defined for each nonstandard space–time point whose smeared operator $\varphi(f)$ defined by (36) satisfy the hyperfinite Wightman axioms. However this has not been proved.

Example 5 (hyperfinite Klein–Gordon field). Let $(a, D_{\mathcal{H}}, D_{\mathcal{K}})$ be the representation of the CCR in Example 4 with $\epsilon = 1$. Set $J = \mathcal{R}^3 := \{-N\epsilon_{\mathcal{R}}, (-N+1)\epsilon_{\mathcal{R}}, \ldots, N\epsilon_{\mathcal{R}}\}^3$ where $N \in {}^\star\mathbf{N}_\infty$, $\epsilon_{\mathcal{R}} \in \mathbf{R}_0^+$ and $N\epsilon_{\mathcal{R}} \in {}^\star\mathbf{R}_\infty$. For $\mathbf{p} \in \mathcal{R}^3$, let $a(\mathbf{p}) = a(f_{\mathbf{p}})$ where $f_{\mathbf{p}}(\mathbf{q}) := \delta_{\mathbf{pq}}$. We see $a(f) = \sum_{\mathbf{p}\in\mathcal{R}^3} \overline{f(\mathbf{p})}a(\mathbf{p})$. For $x \in {}^\star\mathbf{R}$ define $\varphi(x)$ by

$$\varphi(x) = \sqrt{\pi\epsilon_{\mathcal{R}}^3} \sum_{\mathbf{p}\in\mathcal{R}^3} \omega_{\mathbf{p}}^{-1/2}\left[e^{-ip^\mu x_\mu}a(\mathbf{p})^\star + e^{ip^\mu x_\mu}a(\mathbf{p})\right] \tag{37}$$

where $\omega_{\mathbf{p}} := \sqrt{||\mathbf{p}||^2 + m^2}$ and $p = (\omega_{\mathbf{p}}, \mathbf{p})$. For $f \in {}^\star\mathcal{S}(\mathbf{R}^4)$, define $\varphi(f)$ by

$$\varphi(f) = {}^\star\!\int \varphi(x)f(x)\,dx. \tag{38}$$

We see

$$\varphi(f) = \sqrt{\pi\epsilon_{\mathcal{R}}^3} \sum_{\mathbf{p}\in\mathcal{R}^3} \omega_{\mathbf{p}}^{-1/2}[\tilde{f}(p)a(\mathbf{p})^* + \tilde{f}(-p)a(\mathbf{p})] \tag{39}$$

where $\tilde{f}$ is the internal Fourier transform of f.

Note the following fact:

Lemma 5.8. *For any $h \in \mathcal{S}(\mathbf{R}^n)$,*

$$\varepsilon_{\mathcal{R}}^n \sum_{x\in\mathcal{R}^n} {}^\star h(x) \approx \int_{\mathbf{R}^n} h(x)\,dx. \tag{40}$$

Using this Lemma, we have the following.

Theorem 5.9 *(four-dimensional commutation relation).* *For any $\xi \in D_{\mathcal{H}}$ and $f, g \in {}^\sigma\mathcal{S}(\mathbf{R}^4)$,*

$$[\varphi(f), \varphi(g)]\xi \approx \left[\frac{1}{i}\int_{\mathbf{R}^4}\int_{\mathbf{R}^4} \Delta_m(x-y)f(x)g(y)\,dx\,dy\right]\xi, \tag{41}$$

where $\Delta_m(x)$ is the Pauli–Jordan function defined by

$$\Delta_m(x) = \frac{i}{2(2\pi)^3}\int [e^{-ipx} - e^{ipx}]\frac{d\mathbf{p}}{\omega_{\mathbf{p}}}. \tag{42}$$

Proof: Notice the equation

$$\int_{\mathbf{R}^4}\int_{\mathbf{R}^4} \Delta_m(x-y)f(x)g(y)\,dx\,dy = \pi i\int \tilde{f}(-p)\tilde{g}(p) - \tilde{f}(p)\tilde{g}(-p)\frac{d\mathbf{p}}{\omega_{\mathbf{p}}}. \tag{43}$$

We see

$$\pi\epsilon_{\mathcal{R}}^3 \sum_{\mathbf{p},\mathbf{q}\in\mathcal{R}^3} \frac{1}{\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{q}}}}[a(\mathbf{p})\tilde{f}(-p), a(\mathbf{q})\tilde{g}(-q)]\xi = 0$$

$$\pi\epsilon_{\mathcal{R}}^3 \sum_{\mathbf{p},\mathbf{q}\in\mathcal{R}^3} \frac{1}{\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{q}}}}[a^*(\mathbf{p})\tilde{f}(p), a(\mathbf{q})\tilde{g}(-q)]\xi = \pi\epsilon_{\mathcal{R}}^3 \sum_{\mathbf{p}\in\mathcal{R}^3} \frac{1}{\omega_{\mathbf{p}}}\tilde{f}(p)\tilde{g}(-p)\xi$$

etc. Thus, by Lemma 5.8, we have

$$\begin{aligned}[\varphi(f), \varphi(g)]\xi &= \pi\epsilon_{\mathcal{R}}^3 \sum_{\mathbf{p}\in\mathcal{R}^3} \omega_{\mathbf{p}}^{-1}[\tilde{f}(-p)^\star\tilde{g}(p) + \tilde{f}(p)^\star\tilde{g}(-p)]\xi \\ &\approx \left[\frac{1}{i}\int_{\mathbf{R}^4}\int_{\mathbf{R}^4} \Delta_m(x-y) f(x)g(y)\,dx\,dy\right]\xi \quad \square\end{aligned}$$

Corollary *(microscopic causality).* *Let* $f, g \in {}^\sigma\mathcal{S}(\mathbf{R}^4)$. *If* supp$f$ *and* supp$g$ *are space-like separated,*

$$[\varphi(f), \varphi(g)]\xi \approx 0 \tag{44}$$

for any $\xi \in D_{\mathcal{H}}$.

Theorem 5.10 *(two-point Wightman function).*

$$w_2(f, g) := \langle\Omega, \varphi(f)\varphi(g)\Omega\rangle \approx \frac{1}{i}\int_{\mathbf{R}^4}\int_{\mathbf{R}^4} \Delta_m^{(+)}(x-y) f(x)g(y)\,dx\,dy, \tag{45}$$

where

$$\Delta_m^{(+)}(x) = \frac{i}{2(2\pi)^3}\int e^{-ipx}\frac{d\mathbf{p}}{\omega_{\mathbf{p}}}. \tag{46}$$

Proof: Notice the equation

$$\int_{\mathbf{R}^4}\int_{\mathbf{R}^4} \Delta_m^{(+)}(x-y) f(x)g(y)\,dx\,dy = \pi i \int \tilde{f}(-p)\tilde{g}(p)\frac{d\mathbf{p}}{\omega_{\mathbf{p}}}. \tag{47}$$

By Lemma 5.8, we have

$$\begin{aligned}\langle\Omega, \varphi(f)\varphi(g)\Omega\rangle &= \pi\epsilon_{\mathcal{R}}^3 \sum_{\mathbf{p}\in\mathcal{R}^3} \omega_{\mathbf{p}}^{-1}\tilde{f}(-p)\tilde{g}(p) \\ &\approx \frac{1}{i}\int_{\mathbf{R}^4}\int_{\mathbf{R}^4} \Delta_m^{(+)}(x-y) f(x)g(y)\,dx\,dy \quad \square\end{aligned}$$

Theorem 5.11 *(n-point Wightman function).* *If $n \in \mathbf{N}$ and $f_1, f_2, \ldots, \in {}^\sigma\mathcal{S}(\mathbf{R}^4)$,*

$$w_{2n-1}(f_1, \ldots, f_{2n-1}) := \langle \Omega, \varphi(f_1) \cdots \varphi(f_{2n-1})\Omega \rangle \approx 0,$$
$$w_{2n}(f_1, \ldots, f_{2n}) := \langle \Omega, \varphi(f_1) \cdots \varphi(f_{2n})\Omega \rangle$$
$$\approx \sum_{\text{comb}} w_2(f_{i_1}, f_{j_1}) \cdots w_2(f_{i_1}, f_{j_1}) \qquad (48)$$

where $\sum_{\text{comb}}$ denotes the sum over $i_1, j_1, \ldots, i_n, j_n$ such that $1 \leq i_k < j_k \leq 2_n$ $(k = 1, \ldots, n)$, $i_1 < \cdots < i_n$, $i_k \neq j_l$ $(k, l = 1, \ldots, n)$.

Proof: By the calculations similar to the case of the standard field operators, using (45). □

Theorem 5.12. *Let $\mathcal{A}_\mathcal{S}$ be the algebra generated by $\{1, \varphi({}^\star f) \mid f \in \mathcal{S}(\mathbf{R}^4)\}$. Let $D = \mathcal{A}_\mathcal{S}\Omega$. Then the standardized field $\phi(\cdot)$ defined by $\phi(f) = \widehat{\varphi({}^\star f)}_D$ is unitary equivalent to the standard Klein–Gordon field operator (Reed and Simon, 1975), which satisfies the Gårding–Wightman axioms.*

Proof: By Theorem 5.10 and 5.11, the Wightman functions of the standardized field ϕ coincides with those of the standard Klein–Gordon field theory. Thus, by the Wightman reconstruction theorem, the proof is completed. □

Let $N(\mathbf{p}) = a(\mathbf{p})^* a(\mathbf{p})$, $P_0 = \sum_{\mathbf{p} \in \mathcal{R}^3} \omega_\mathbf{p} N(\mathbf{p})$ and $P_\alpha = \sum_{\mathbf{p} \in \mathcal{R}^3} \mathbf{P}_\alpha N(\mathbf{p})$ for $\alpha = 1, 2, 3$.

Theorem 5.13. $(\varphi, \Omega, (P_0, \ldots, P_3))$ *satisfies the hyperfinite Wightman axioms.*

Proof: HW1 is seen from $\mathcal{A}_\mathcal{S} \subset \mathcal{A}$ ($\mathcal{A}$ is defined in Example 4). HW2 is clear. HW3 is seen from the S-continuity of the CCR representation $(a, D_\mathcal{H}, D_\mathcal{K})$. HW4 follows from Theorem 5.11. To see HW5, check

$$e^{ia^\mu P_\mu} a(\mathbf{p}) e^{-a^\mu P_\mu} = e^{-ia^\mu P_\mu} a(\mathbf{p}), \qquad (49)$$

and we get

$$e^{ia^\mu P_\mu} \varphi(x) e^{-a^\mu P_\mu} = \varphi(x - a). \qquad (50)$$

Therefore, we see that HW5 is satisfied. We also find

$$P^\mu P_\mu = \sum_\mathbf{p} p^\mu p_\mu N(\mathbf{p}) = \sum_\mathbf{p} m^2 N(\mathbf{p}) \geq 0. \qquad (51)$$

Therefore HW6 is satisfied. HW7 follows from Corollary of Theorem 5.9. HW8 is seen from $P_0\xi = 0$ iff $\xi = c\Omega$. □

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